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Acoustic radiation force due to incident plane-progressive waves on coated spheres immersed in ideal fluids

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Abstract. In this study, the acoustic radiation force resulting from the interaction of a plane progressive wave with a coated sphere was examined. The linear acoustic scattering problem was obtained first by solving the classical boundary conditions to obtain the required coefficients. The radiation force was then determined by averaging the momentum flux tensor expressed in terms of the total scattering pressure or velocity potential in an ideal fluid. Numerical calculations of the radiation force function Y_p, which is the radiation force per unit energy density and unit cross-section, were displayed versus the dimensionless size parameter $x = k_1 b$ (k_1 is the wave number in the exterior fluid and b the radius of the uncoated sphere) over a large range of frequencies. Particular emphasis has been focused on the coating thickness and the absorption of sound inside the outer covering layer. The fluid-loading effect on the radiation force function curves was also analysed.

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1 Introduction

A problem of interest in the discussion of the nonlinear interaction of sound waves with objects is the application of a force (known as the acoustic radiation force) due to the transfer of the momentum flux [1]. Moreover, intrinsically the nature of wave motion in fluids results in a variety of nonlinear effects that compete with the radiation force, such as streaming. Since the work on the acoustic radiation force on a rigid sphere in an axisymmetric wave field by King [2], the radiation force on a sphere with different mechanical properties (compressible, fluid, elastic) has been investigated [3–8]. It has been proved theoretically and experimentally that there appears a series of maxima and minima in the radiation force function curves which correspond to the resonance frequencies of the sphere's elastic vibrations.

Furthermore, various aspects of the acoustic radiation force on rigid spheres were examined with particular attention given to dissipation effects, such as viscous and thermal losses [9–12]. In the presence of viscosity, one should distinguish between the two notions: that of the mean acoustic radiation force exerted by a sound field on an object, and that of the total radiation force (known as "radiation pressure") which is the rate at which the mean momentum transported by the sound wave changes because of the disturbance due to the presence of the object along the wave's path. Only part of it contributes to the mean force acting on the object, while another part is transferred into the fluid to produce acoustic streaming with dipole-symmetry [12].

The sustained interest in these problems is due to the importance of scattering and attenuation in many areas of research such as acoustic levitation, contrast agent imaging, and other medical applications. One of the important applications of the radiation force on spheres is the determination of acoustic intensity from the radiation force measurement for the calibration of high frequency transducers [13].

Although the single sphere results are useful, the real interest, as far as the previously mentioned applications are concerned, is concentrated on coated scatterers. For example, biological cells involve a central nucleus coated by a cytoplasmic layer, and non-destructive evaluation usually concerns the identification of core media within coated objects. Human organs can also be seen as scattering obstacles buried within the body, and air or water polluants are covered by all kinds of substances. Bioactive layered spheres are also considered to be potentially excellent for delivering drugs to localized targeting sites, such as mass lesions or tumors. Moreover, coated spheres prolong the circulation time in blood vessels, preventing the drugs from dissolution before reaching the desired target. In such a process, it is crucial to have a priori knowledge of the radiation force on such bioactive layered spheres in order to non-destructively guide them to the desired location.

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The goal of the present work is to develop the theory of the acoustic radiation force acting on coated spheres and placed in a plane progressive sound field. The coated spheres were assumed to be submerged in compressible nonviscous fluids. Numerical calculations are performed to show how the acoustic radiation force can be affected by absorption characteristics, thickness of the coating layer, and variations of the sphere's mechanical parameters. The viscoelastic layer is assumed to be ideally bonded to the sphere. Furthermore, it is shown that the fluid-loading drastically influences the radiation force, resulting in new interesting effects.

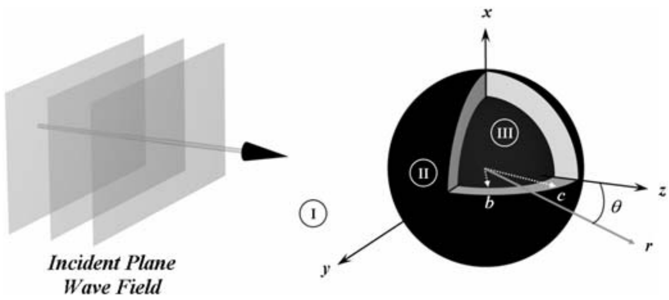

Fig. 1. A coated sphere placed in a progressive plane-wave field incident from the direction $\theta = \pi$.

2 Method

The acoustic radiation force on a coated sphere can be calculated by integrating the time-average radiation-stress tensor over the sphere's surface. The radiation-stress tensor is expressed in terms of the total (incident and scattered) linear acoustic potential velocity or pressure fields. It is therefore essential to calculate the linear acoustic scattering field disturbed by the coated sphere for the purpose of computing the radiation force.

2.1 Acoustic scattering by a coated sphere

In this section, the general problem of acoustic scattering from a layered sphere immersed in a nonviscous fluid is considered. The geometry and the coordinate system used are shown in Figure 1. The center of the layered sphere coincides with the origin of a rectangular coordinate system, and the plane waves approach the sphere along the negative z-axis.

In the exterior fluid medium (medium 1), the linearized continuity and Euler's equations can be written as [14]

$$\frac{\partial \rho_1}{\partial t} + \rho_1 \nabla \cdot \mathbf{v} = 0, \tag{1}$$

$$\rho_1 \frac{\partial \mathbf{v}}{\partial t} + \nabla P = 0. \tag{2}$$

ρ_1 is the mass density, P is the ambient pressure equal to P_0 in the absence of sound, and $\mathbf{v}$ is the fluid velocity. For an ideal (nonviscous) fluid, the linearized equation of state is $P = c_1^2 \rho'$, where c_1 is the speed of sound, and ρ' is the density that takes care of the acoustic compression of the medium during the passage of the sound wave. Equations (1) and (2) can be combined to a single equation for the velocity $\mathbf{v}$ such as:

$$\frac{\partial^2 \mathbf{v}}{\partial t^2} = c_1^2 \nabla \left(\nabla \cdot \mathbf{v} \right). \tag{3}$$

Assuming that the velocity $\mathbf{v}$ can be derived from a scalar potential φ_1 such as $\mathbf{v} = -\nabla \varphi_1$, equation (3) can be rewritten in an equivalent form:

$$\frac{\partial^2 \varphi_1}{\partial t^2} = c_1^2 \nabla^2 \varphi_1. \tag{4}$$

Assuming also that the incident field is composed of monochromatic plane waves, the solutions of equation (4) are of the form:

$$\varphi_1 \left(r, \theta, t\right) = \mathrm{Re}\left[\varphi_1 \left(r, \theta, \omega\right) e^{-i\omega t}\right], \tag{5}$$

where Re indicates the real part of a complex number, and $\varphi_1 \left(r, \theta, \omega\right)$ may be complex. Replacing equation (5) in equation (4), and after some manipulation, the Helmholtz equation is obtained:

$$\left(\nabla^2 + k_1^2\right) \varphi_1 = 0, \tag{6}$$

where the compressional wave number in the fluid is $k_1 = \omega / c_1$.

Therefore, the *total* scalar velocity potential field (solution of Eq. (6)) is the sum of the incident and scattered fields that can be expressed in spherical coordinates by:

$$\varphi_1 \left(r, \theta\right) = \Phi_0 \sum_{n=0}^{\infty} i^n \left(2n+1\right) \Big(j_n \left(k_1 r\right) + A_n h_n^{(1)} \left(k_1 r\right) \Big) P_n \left(\cos\theta\right), \tag{7}$$

where Φ_0 is the amplitude, $j_n(\cdot)$ and $h_n^{(1)}(\cdot)$ are the spherical Bessel and Hankel functions of the first kind of order n, respectively, k_1 is the wave number in the exterior fluid medium (medium 1), A_n are the unknown scattering coefficients that will be determined by the appropriate boundary conditions, and $P_n(.)$ are Legendre polynomials.

The waves inside the layered sphere (media 2 and 3) will be represented by suitable solutions of the equation of motion of a solid elastic medium (since no absorption is included yet), which may be written [15]:

$$(\lambda_{2,3}+2\mu_{2,3})\nabla(\nabla\cdot\mathbf{U}_{2,3})-\mu_{2,3}\nabla\times(\nabla\times\mathbf{U}_{2,3})=\rho_{2,3}\frac{\partial^2\mathbf{U}_{2,3}}{\partial t^2}, \tag{8}$$

where $\lambda_{2,3}$ and $\mu_{2,3}$ are the Lamé coefficients, and $\rho_{2,3}$ the mass densities for the covering layer (medium 2) and core material (medium 3), respectively. $\mathbf{U}_{2,3}$ is the vector displacement that can be expressed as a sum of the gradient of a scalar potential $\Phi_{2,3}$ and the curl of a vector potential $\mathbf{\Psi}_{2,3}$ as follows:

$$\mathbf{U}_{2,3} = \nabla \Phi_{2,3} + \left(\nabla \times \mathbf{\Psi}_{2,3}\right). \tag{9}$$

Using the problem symmetry, the vector potential $\mathbf{\Psi}_{2,3}$ reduces to a scalar equation; i.e. $\mathbf{\Psi}_{2,3} = (0, 0, \Psi_{2,3})$, and using the condition $\nabla \cdot \mathbf{\Psi}_{2,3} = 0$, the Helmholtz equations for the solid medium are obtained as follows:

$$\left(\nabla^2 + k_{L,2,3}^2\right) \Phi_{2,3} = 0, \tag{10}$$

$$\left(\nabla^2 + k_{S,2,3}^2\right) \Psi_{2,3} = 0, \tag{11}$$

where $k_{L,2,3} = \frac{\omega}{[(\lambda_{2,3}+2\mu_{2,3})/\rho_{2,3}]^{1/2}}$, and $k_{S,2,3} = \frac{\omega}{[\mu_{2,3}/\rho_{2,3}]^{1/2}}$, refer to the longitudinal and shear wave numbers in the solid media, respectively.

Similarly, the longitudinal and shear waves inside the layer (medium 2), are represented in spherical coordinates by:

$$\Phi_2(r,\theta) = \Phi_0 \sum_{n=0}^{\infty} i^n (2n+1) \Big(B_n j_n (k_{L,2} r) + C_n n_n (k_{L,2} r) \Big) P_n (\cos\theta), \tag{12}$$

$$\Psi_2(r,\theta) = \Phi_0 \sum_{n=0}^{\infty} i^n (2n+1) \Big(D_n j_n (k_{S,2} r) + E_n n_n (k_{S,2} r) \Big) P_n (\cos\theta). \tag{13}$$

where $n_n(\cdot)$ are the spherical Bessel functions of the second kind. Sound absorption by the viscoelastic layer is modeled by introducing complex size parameters (or wave numbers, respectively), accounting for losses inside the covering layer. Incorporating complex wave numbers into the acoustic scattering theory holds only for *linear* viscoelasticity [16]. Here, it is assumed that the normalized absorption coefficients of compressional and shear waves are constant quantities independent of frequency.

In the core material (medium 3), the potentials solution of equations (10) and (11) are given by

$$\Phi_3(r,\theta) = \Phi_0 \sum_{n=0}^{\infty} i^n (2n+1) F_n j_n (k_{L,3} r) P_n(\cos\theta), \tag{14}$$

$$\Psi_3(r,\theta) = \Phi_0 \sum_{n=0}^{\infty} i^n (2n+1) G_n j_n (k_{S,3} r) P_n(\cos\theta), \tag{15}$$

A_n, B_n, C_n, D_n, E_n, F_n, and G_n, are the unknown coefficients determined from the following boundary conditions:

- At the outside boundary of the coated sphere (interface at medium 1 and 2), the displacements (velocities) and normal stresses must be continuous and the tangential stresses must be zero, leading to:
  - $v_r^{(1)}\Big|_{r=c} = -i\omega\, U_r^{(2)}\Big|_{r=c}$;
  - $\sigma_{rr}^{(1)}\Big|_{r=c} = \sigma_{rr}^{(2)}\Big|_{r=c}$;
  - $\sigma_{r\theta}^{(2)}\Big|_{r=c} = 0.$
- At the interface between the outer layer and core material (interface at medium 2 and 3), the radial and tangential displacements are continuous, and the radial and tangential stresses of adjoining materials are equal:
  - $U_r^{(2)}\Big|_{r=b} = U_r^{(3)}\Big|_{r=b}$;
  - $U_\theta^{(2)}\Big|_{r=b} = U_\theta^{(3)}\Big|_{r=b}$;
  - $\sigma_{rr}^{(2)}\Big|_{r=b} = \sigma_{rr}^{(3)}\Big|_{r=b}$;
  - $\sigma_{r\theta}^{(2)}\Big|_{r=b} = \sigma_{r\theta}^{(3)}\Big|_{r=b}$.

The detailed expressions of the velocities, displacements and stress components are given in Appendix A.

The boundary conditions lead to seven linear equations. The general solution for A_n is given by

$$A_n = \frac{\begin{vmatrix} \lambda_1^* & \lambda_{12} & \lambda_{13} & \lambda_{14} & \lambda_{15} & 0 & 0 \\ \lambda_2^* & \lambda_{22} & \lambda_{23} & \lambda_{24} & \lambda_{25} & 0 & 0 \\ 0 & \lambda_{32} & \lambda_{33} & \lambda_{34} & \lambda_{35} & 0 & 0 \\ 0 & \lambda_{42} & \lambda_{43} & \lambda_{44} & \lambda_{45} & \lambda_{46} & \lambda_{47} \\ 0 & \lambda_{52} & \lambda_{53} & \lambda_{54} & \lambda_{55} & \lambda_{56} & \lambda_{57} \\ 0 & \lambda_{62} & \lambda_{63} & \lambda_{64} & \lambda_{65} & \lambda_{66} & \lambda_{67} \\ 0 & \lambda_{72} & \lambda_{73} & \lambda_{74} & \lambda_{75} & \lambda_{76} & \lambda_{77} \end{vmatrix}}{\begin{vmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} & \lambda_{15} & 0 & 0 \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24} & \lambda_{25} & 0 & 0 \\ 0 & \lambda_{32} & \lambda_{33} & \lambda_{34} & \lambda_{35} & 0 & 0 \\ 0 & \lambda_{42} & \lambda_{43} & \lambda_{44} & \lambda_{45} & \lambda_{46} & \lambda_{47} \\ 0 & \lambda_{52} & \lambda_{53} & \lambda_{54} & \lambda_{55} & \lambda_{56} & \lambda_{57} \\ 0 & \lambda_{62} & \lambda_{63} & \lambda_{64} & \lambda_{65} & \lambda_{66} & \lambda_{67} \\ 0 & \lambda_{72} & \lambda_{73} & \lambda_{74} & \lambda_{75} & \lambda_{76} & \lambda_{77} \end{vmatrix}}, \tag{16}$$

where λ_i^* and λ_{ij} are the dimensionless elements of the determinants given in Appendix B.

2.2 Acoustic radiation force on a layered spherical shell

The general theory of the acoustic radiation force on a boundary moving with a velocity of the first order was developed by Yosioka and Kawasima [3] and is expressed as

$$\langle \mathbf{F} \rangle = -\iint_S \left[\left(\frac{1}{2}\frac{\rho_1}{c_1^2} \left\langle \left(\frac{\partial \varphi_1}{\partial t} \right)^2 \right\rangle - \frac{1}{2}\rho_1 \left\langle \left| \boldsymbol{\nabla}\varphi_1 \right|^2 \right\rangle \right) \mathbf{n} + \rho_1 \left\langle (v_n \mathbf{n} + v_t \mathbf{t}) v_n \right\rangle \right] dS, \tag{17}$$

where φ_1 is the total field given by equation (7), ρ_1 and c_1 are the mass density and sound velocity in the exterior fluid medium (medium 1), respectively, $v_n\mathbf{n}$ and $v_t\mathbf{t}$ are the normal and tangential components of the particle velocity of the boundary, respectively, and the symbol $\langle . \rangle$ denotes the time-averaging.

According to Hasegawa and Yosioka [7], in the direction of wave propagation (z-direction) the value $\langle F_z \rangle$ of the radiation force $\langle \mathbf{F} \rangle$ is expressed as

$$\langle F_z \rangle = \langle F_r \rangle + \langle F_\theta \rangle + \langle F_{r\theta} \rangle + \langle F_t \rangle, \tag{18}$$

Table 1. Material parameters used in the numerical calculations.

Material	Mass density $[10^3\ \mathrm{kg/m^3}]$	Compressional Velocity [m/s]	Shear Velocity [m/s]	Normalized longitudinal absorption γ_{21}	Normalized shear absorption γ_{22}
Gold	19.3	3240	1200	...	...
Stainless Steel	7.9	5240	2978	...	...
Phenolic polymer	1.22	2840	1320	0.0119	0.0257
Mercury	13.6	1407	...	...	...
Water	1.00	1500	...	...	...

where

$$\langle F_r \rangle = -\pi c^2 \rho_1 \left\langle \int_0^{\pi} \left(\frac{\partial \varphi_1}{\partial r} \right)^2_{r=c} \sin\theta \cos\theta \, d\theta \right\rangle,$$
$$\langle F_\theta \rangle = \pi \rho_1 \left\langle \int_0^{\pi} \left(\frac{\partial \varphi_1}{\partial \theta} \right)^2_{r=c} \sin\theta \cos\theta \, d\theta \right\rangle,$$
$$\langle F_{r\theta} \rangle = 2\pi c \rho_1 \left\langle \int_0^{\pi} \left(\frac{\partial \varphi_1}{\partial r} \right)_{r=c} \left(\frac{\partial \varphi_1}{\partial \theta} \right)_{r=c} \sin^2\theta \, d\theta \right\rangle,$$
$$\langle F_t \rangle = -\frac{\pi c^2 \rho_1}{c_1^2} \left\langle \int_0^{\pi} \left(\frac{\partial \varphi_1}{\partial t} \right)^2_{r=c} \sin^2\theta \cos\theta \, d\theta \right\rangle. \quad (19)$$

The final expression of the total force can be represented by

$$\langle F_z \rangle = \langle E \rangle \, S_c Y_p, \quad (20)$$

where $\langle E \rangle = \frac{1}{2}\rho_1 k_1^2 \left| \Phi_0 \right|^2$ is the mean energy density of the incident plane acoustic wave field, and $S_c = \pi c^2$ is the cross-sectional area. Y_p is a dimensionless factor called the radiation force function, which depends on the scattering and absorption properties of the target and is the radiation force per unit cross section and unit energy density.

After replacing equation (7) in equation (18) using equation (19), Y_p can be expressed as

$$Y_p = -\frac{4}{(k_1 c)^2} \sum_{n=0}^{\infty} (n+1) \times \left[\alpha_n + \alpha_{n+1} + 2\left(\alpha_n \alpha_{n+1} + \beta_n \beta_{n+1} \right) \right], \quad (21)$$

where α_n and β_n are real and imaginary parts of the scattering coefficients A_n defined by equation (16).

3 Numerical results and discussion

In order to illustrate the nature and general behavior of the solution, a numerical example is considered. The objective is to display the computed radiation force functions of the coated sphere as a function of the size parameter $x = k_1 b$. In this case, the outer layer comprises a phenolic polymer, a highly absorbent material, and the core medium comprises gold material which is considered to be lossless. Another example is also considered in which the core material was chosen to be stainless steel. The materials' mechanical parameters are listed in Table 1.

Absorption of sound inside the outer covering layer is incorporated in the theory by introducing complex size parameters. The normalized absorption coefficients for both compressional and shear waves are listed in Table 1. The outer covering layer has a thickness $e_1 = c/b$ (see Fig. 1).

Computations for gold and stainless steel spheres covered by a phenolic polymer were performed over a large range of frequencies corresponding to $0 \le x \le 60$ with intervals of 0.01. It is very important to choose a sufficiently small sampling increment since resonance peaks are very sharp and improper sampling may lead to false curves. It was verified that by reducing the sampling increment, no prominent changes were perceived in the curves. It is crucial also to choose a relatively large number in the series (i.e. Eq. (21)) at least of the same order of $x = k_1 b$ so as to ensure a proper convergence. In our case, the number N in the series was fixed to twice the maximum size parameter bandwidth (i.e. $N = 120$).

As an initial test, the calculations were performed for uncoated ($e_1 = 1$), gold (Fig. 2a) and stainless steel (Fig. 3a) spheres immersed in water. This was done in order to compare the results with those of Anson and Chivers [17] and Hasegawa, et al. [18]. Excellent agreement was found.

In the following, the absorptive behavior of the coating is investigated and the radiation force function curves were plotted for different thicknesses of the outer covering, with and without absorption.

Figures 2b–d display the variation of the radiation force function for layered gold spheres with and without absorption. It is obvious here that the (absorptive) coating affects the resonance frequencies of the metallic sphere. The main observation is the damping of resonance peaks when the thickness of the outer covering increases. However, one notices the enhancement in the radiation force function curve for a thick absorptive layer while $x = k_1 b$ increases ($x > 10$) (Fig. 2d). This enhancement is also observed for stainless steel material (Fig. 3d). One important comment could be said in order to give a full interpretation of this enhancement at high size parameter values (high x) that is related to sound-energy absorption; when absorption is strong, the sound-energy density in the area of the incident plane wave field is high

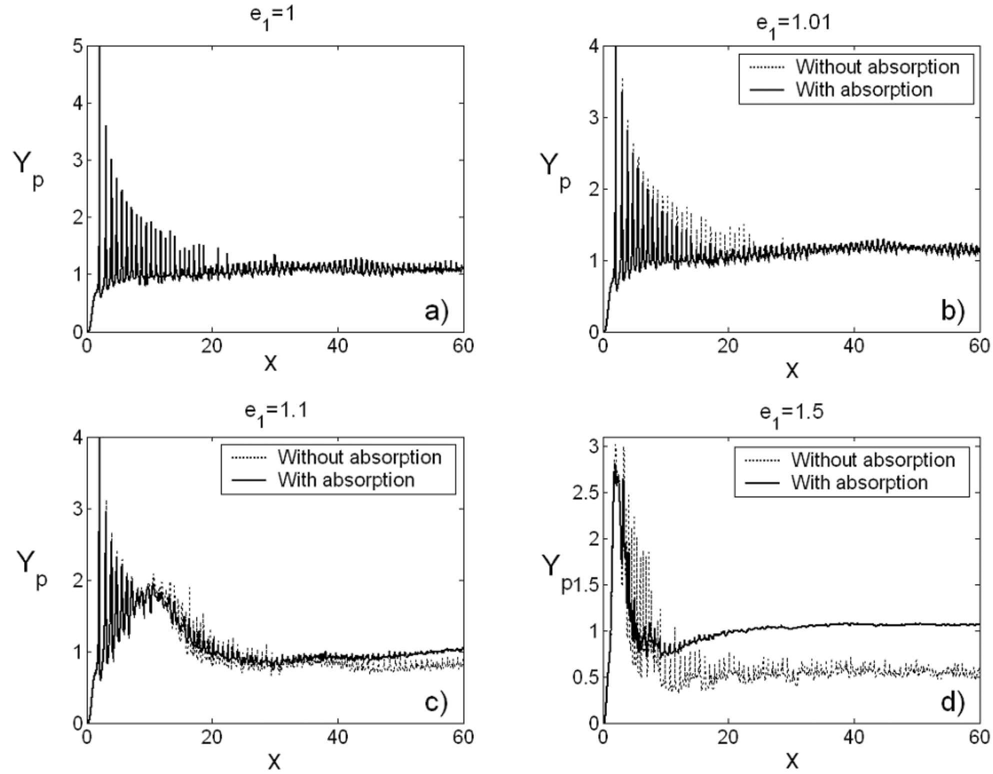

Fig. 2. Y_p curves for phenolic polymer-coated gold spheres immersed in water for different thicknesses (e_1) of the covering layer with and without absorption. $e_1 = 1$ corresponds to uncoated spheres.

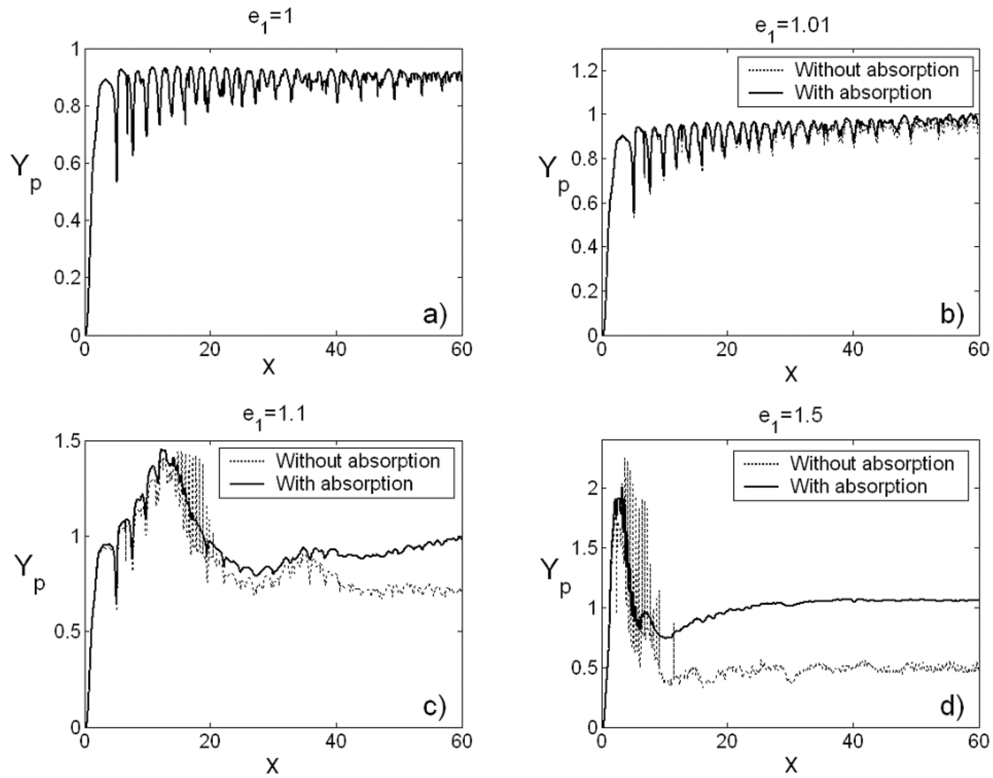

Fig. 3. The same as in Figure 2 but for phenolic polymer-coated stainless steel spheres.

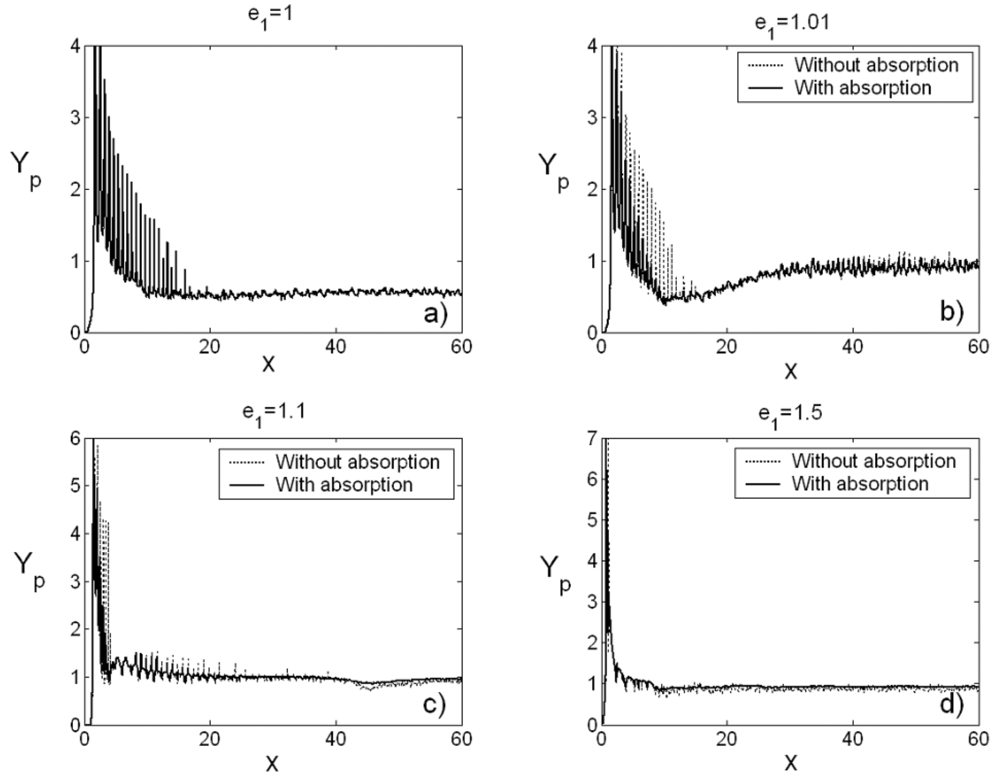

Fig. 4. The Y_p curves for phenolic polymer-coated gold spheres immersed in mercury for different thicknesses of the covering layer with and without absorption.

compared to the case without absorption. Hence, according to equation (20), the net force per cross-section acting on the coated sphere in the direction of the incident waves is high, which is confirmed by these results.

Additional calculations were performed in order to study the fluid-loading effect on the radiation force. Figures 4b−d show Y_p curves for gold coated spheres immersed in a high density fluid (in this case mercury). The density of gold is higher than that of mercury, and this was shown to have a minor effect on the radiation force for uncoated spheres (Fig. 2a versus Fig. 4a). However, as long as the thickness of the outer covering increases, a series of resonance peaks tends to disappear (Fig. 4b−d), and the enhancement in the radiation force function curves initially observed in Figure 2d (while x increases) has been removed. However, a low frequency ($x < 3$) enhancement in the radiation force function curves is observed. The main cause is that increased fluid-loading produces interactions between various resonance modes which can have a significant effect on the modal spectrum of the sphere. Studies on sound scattering from absorbing cylinders have shown similar behavior [19].

Figures 5b–d display Y_p curves for stainless steel coated spheres immersed in mercury. By comparing Figure 3a with Figure 5a, it is obvious that the fluid-loading drastically changes the radiation force, since the stainless steel material is known to have a lower density than mercury. The enhancement at low frequency in the radiation force function curves is still noticeable for this material, especially when the thickness of the outer covering increases (Fig. 5d). From these figures, it is concluded that the radiation force is not greatly affected by sound (or ultrasound) absorption while x increases ($x > 10$).

The positions of extrema in the figures correspond to resonance frequencies of the sphere vibrations. A detailed discussion on whether the resonances are manifested as either maxima or minima in the Y_p curves is very subtle. Whether these resonances appear as dips instead of peaks in these curves is strongly related to the sphere's and coating material mechanical properties, and the surrounding medium. Further studies should seek to identify each factor that contributes to the acoustic radiation force separately in order to address that question.

4 Conclusion

In this paper, we have addressed the problem of the acoustic radiation force on a spherically shaped target. Although the problem has received intense treatment in the literature, it has required some additional analysis to take into account more general fields. This work presents analytical solutions as well as numerical results for the acoustic radiation force arising from the interaction of a plane progressive sound wave with an elastic sphere that is coated by a viscoelastic sound-absorptive material and immersed in an inviscid fluid. The prime objective was to investigate the effect of the outer covering on the radiation force. Moreover, sound absorption by the outer

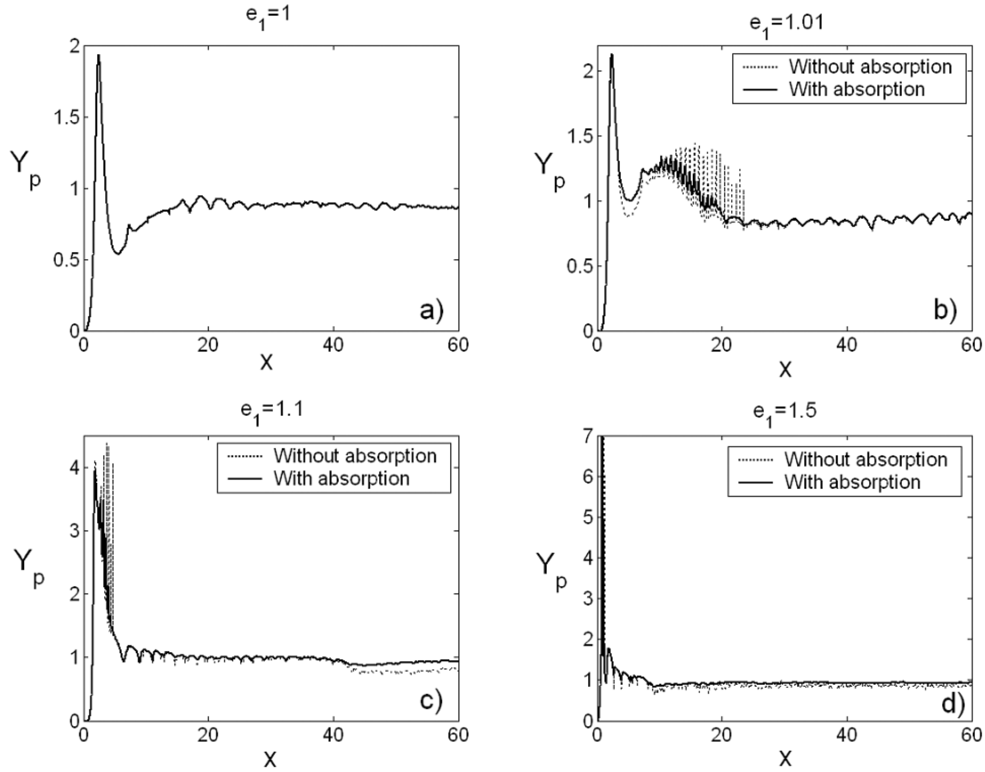

Fig. 5. The same as in Figure 4 but for phenolic polymer-coated stainless steel spheres. One notices the high resonance peak at low frequency ($x = 0.88$) especially when the thickness of the viscoelastic layer increases (Fig. d).

covering was also examined as well as the fluid-loading effect. Numerical results reveal the important consequences of the inclusion of absorption inside the viscoelastic layer, such as the acoustic radiation force enhancement when the outer covering thickness increases. Furthermore, the fluid-loading effect has shown to produce interactions between various vibrational modes of the layered sphere and alter the position of their relative resonance peaks or dips, resulting in a low frequency ($0 \leq x \leq 3$) enhancement in the radiation force function curves. The proposed model leads to an extension of the standard theory on the acoustic radiation force experienced by elastic spheres since its relative results are obtained here by simply allowing $e_1 = 1$. The results for spherical shells can also be obtained in a straightforward manner by considering the sphere as a fluid medium (shear velocity equal to zero) coated by elastic or a viscoelastic layer [20]. A possible application of the theory is for the estimation of the covering layer thickness by inverting the problem arising from the radiation force function curves.

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Appendix A: Field equations

The basic field equations in spherical coordinates are given as follows; the velocity component of the wave in the exterior fluid medium is $v_r^{(1)} = -\frac{\partial \varphi_1}{\partial r}$, where φ_1 is given by equation (7).

Similarly, the displacements expressed in terms of potentials in the layered sphere are:

$$U_r^{(2,3)} = \frac{\partial \Phi_{2,3}}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial \left(\mathbf{\Psi}_{2,3} \sin\theta\right)}{\partial \theta},$$

$$U_\theta^{(2,3)} = \frac{1}{r} \frac{\partial \mathbf{\Phi}_{2,3}}{\partial \theta} - \frac{1}{r} \frac{\partial \left(r \mathbf{\Psi}_{2,3}\right)}{\partial \theta},$$

where $\Phi_{2,3}$ and $\mathbf{\Psi}_{2,3}$ are the scalar and vector potentials given in equations (12–15).

The stress component in the exterior fluid is $\sigma_{rr}^{(1)} = i\omega\rho_1\varphi_1$, where ρ_1 is the exterior fluid mass density, and the stresses components in the layered sphere are:

$$\sigma_{rr}^{(2,3)} = 2\mu_{2,3} \frac{\partial U_r^{(2,3)}}{\partial r} + \lambda_{2,3} \left(\nabla \cdot \mathbf{U}_{2,3}\right),$$

$$\sigma_{r\theta}^{(2,3)} = \mu_{2,3} \left(\frac{1}{r} \frac{\partial U_r^{(2,3)}}{\partial \theta} + \frac{\partial U_\theta^{(2,3)}}{\partial r} - \frac{U_\theta^{(2,3)}}{r} \right),$$

where $\lambda_{2,3}$ and $\mu_{2,3}$ are the Lamé coefficients and $\mathbf{U}_{2,3}$ is the vector displacement (Eq. (9)).

Appendix B: Matrix elements

ρ_1 and ρ_2 are the mass densities of the fluid surrounding the sphere and the viscoelastic coating, respectively.

$e_1 = c/b$ where c and b are the outer and inner radius (Fig. 1). $x = k_1 b$; where $k_1 = \frac{\omega}{c_1}$ and c_1 is the sound velocity in the fluid medium, $\tilde{x}_{21} = x\frac{c_1}{c_{21}}(1 + i\gamma_{21})$ and $\tilde{x}_{22} = x\frac{c_1}{c_{22}}(1 + i\gamma_{22})$ where c_{21} and c_{22} are the compressional and shear sound velocities in the viscoelastic layer and γ_{21} and γ_{22} their corresponding absorption coefficients (Tab. 1), respectively, $x_{31} = x\frac{c_1}{c_{31}}$ and $x_{32} = x\frac{c_1}{c_{32}}$ where c_{31} and c_{32} are the compressional and shear sound velocities in the core material, respectively. $y_1 = xe_1$, $\tilde{y}_{21} = \tilde{x}_{21}e_1$, $\tilde{y}_{22} = \tilde{x}_{22}e_1$, $y_{31} = x_{31}e_1$, $y_{32} = x_{32}e_1$ and $\Lambda_{23} = \frac{\rho_2}{\rho_3}\left(\frac{c_{22}}{c_{32}}\right)^2$, where ρ_3 is the core material mass density.

The following terms are the expressions for the elements of determinants appearing in equation (16)

$$\lambda_{11} = \frac{\rho_1}{\rho_2}\tilde{y}_{22}^2 h_n^{(1)}(y_1),$$
$$\lambda_{12} = \left(2n(n+1) - \tilde{y}_{22}^2\right) j_n(\tilde{y}_{21}) - 4\tilde{y}_{21} j_n'(\tilde{y}_{21}),$$
$$\lambda_{13} = \left(2n(n+1) - \tilde{y}_{22}^2\right) n_n(\tilde{y}_{21}) - 4\tilde{y}_{21} n_n'(\tilde{y}_{21}),$$
$$\lambda_{14} = 2n(n+1)\left(\tilde{y}_{22} j_n'(\tilde{y}_{22}) - j_n(\tilde{y}_{22})\right),$$
$$\lambda_{15} = 2n(n+1)\left(\tilde{y}_{22} n_n'(\tilde{y}_{22}) - n_n(\tilde{y}_{22})\right),$$
$$\lambda_{21} = -y_1 h_n^{(1)'}(y_1),$$
$$\lambda_{22} = \tilde{y}_{21} j_n'(\tilde{y}_{21}),$$
$$\lambda_{23} = \tilde{y}_{21} n_n'(\tilde{y}_{21}),$$
$$\lambda_{24} = n(n+1) j_n(\tilde{y}_{22}),$$
$$\lambda_{25} = n(n+1) n_n(\tilde{y}_{22}),$$
$$\lambda_{32} = 2\left(j_n(\tilde{y}_{21}) - \tilde{y}_{21} j_n'(\tilde{y}_{21})\right),$$
$$\lambda_{33} = 2\left(n_n(\tilde{y}_{21}) - \tilde{y}_{21} n_n'(\tilde{y}_{21})\right),$$
$$\lambda_{34} = 2\tilde{y}_{22} j_n'(\tilde{y}_{22}) + \left(\tilde{y}_{22}^2 - 2n(n+1) + 2\right) j_n(\tilde{y}_{22}),$$
$$\lambda_{35} = 2\tilde{y}_{22} n_n'(\tilde{y}_{22}) + \left(\tilde{y}_{22}^2 - 2n(n+1) + 2\right) n_n(\tilde{y}_{22}),$$
$$\lambda_{42} = \tilde{x}_{21} j_n'(\tilde{x}_{21}),$$
$$\lambda_{43} = \tilde{x}_{21} n_n'(\tilde{x}_{21}),$$
$$\lambda_{44} = n(n+1) j_n(\tilde{x}_{22}),$$
$$\lambda_{45} = n(n+1) n_n(\tilde{x}_{22}),$$
$$\lambda_{46} = -x_{31} j_n'(x_{31}),$$
$$\lambda_{47} = -n(n+1) j_n(x_{32}),$$
$$\lambda_{52} = -j_n(\tilde{x}_{21}),$$
$$\lambda_{53} = -n_n(\tilde{x}_{21}),$$
$$\lambda_{54} = -\tilde{x}_{22} j_n'(\tilde{x}_{22}) - j_n(\tilde{x}_{22}),$$
$$\lambda_{55} = -\tilde{x}_{22} n_n'(\tilde{x}_{22}) - n_n(\tilde{x}_{22}),$$
$$\lambda_{56} = j_n(x_{31}),$$
$$\lambda_{57} = x_{32} j_n'(x_{32}) + j_n(x_{32}),$$
$$\lambda_{62} = \Lambda_{23}\left(\left(2n(n+1) - \tilde{x}_{22}^2\right) j_n(\tilde{x}_{21}) - 4\tilde{x}_{21} j_n'(\tilde{x}_{21})\right),$$
$$\lambda_{63} = \Lambda_{23}\left(\left(2n(n+1) - \tilde{x}_{22}^2\right) n_n(\tilde{x}_{21}) - 4\tilde{x}_{21} n_n'(\tilde{x}_{21})\right),$$
$$\lambda_{64} = 2n(n+1)\Lambda_{23}\left(\tilde{x}_{22} j_n'(\tilde{x}_{22}) - j_n(\tilde{x}_{22})\right),$$
$$\lambda_{65} = 2n(n+1)\Lambda_{23}\left(\tilde{x}_{22} n_n'(\tilde{x}_{22}) - n_n(\tilde{x}_{22})\right),$$
$$\lambda_{66} = 4x_{31} j_n'(x_{31}) - \left(2n(n+1) - x_{32}^2\right) j_n(x_{31}),$$
$$\lambda_{67} = 2n(n+1)\left(j_n(x_{32}) - x_{32} j_n'(x_{32})\right),$$
$$\lambda_{72} = 2\left(j_n(\tilde{x}_{21}) - \tilde{x}_{21} j_n'(\tilde{x}_{21})\right),$$
$$\lambda_{73} = 2\left(n_n(\tilde{x}_{21}) - \tilde{x}_{21} n_n'(\tilde{x}_{21})\right),$$
$$\lambda_{74} = 2\tilde{x}_{22} j_n'(\tilde{x}_{22}) + \left(\tilde{x}_{22}^2 - 2n(n+1) + 2\right) j_n(\tilde{x}_{22}),$$
$$\lambda_{75} = 2\tilde{x}_{22} n_n'(\tilde{x}_{22}) + \left(\tilde{x}_{22}^2 - 2n(n+1) + 2\right) n_n(\tilde{x}_{22}),$$
$$\lambda_{76} = \frac{2}{\Lambda_{23}}\left(x_{31} j_n'(x_{31}) - j_n(x_{31})\right),$$
$$\lambda_{77} = -\frac{1}{\Lambda_{23}}\left(2x_{32} j_n'(x_{32}) + \left(x_{32}^2 - 2n(n+1) + 2\right) j_n(x_{32})\right),$$
$$\lambda_1^* = -\frac{\rho_1}{\rho_2}\tilde{y}_{22}^2 j_n(y_1),$$
$$\lambda_2^* = y_1 j_n'(y_1).$$

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